Poisson-Furstenberg boundaries, large-scale geometry and growth of groups

Anna Erschler

Abstract. We give a survey of recent results on the Poisson-Furstenberg boundaries of random walks on groups, and their applications. We describe sufficient conditions for random walk to have non-trivial boundary, or, on the contrary, to have trivial boundary. We review recent progress in description of the boundary for random walks on various groups, including wreath products. We describe how the Poisson-Furstenberg boundary can be used to obtain lower bounds for the growth function of the groups of intermediate growth. We also discuss relation between properties of the boundary with other asymptotic properties of groups, including isoperimetry and various characteristics of random walks.

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1. Boundaries of random walks on groups

The Poisson boundary is a probability space, defined by a Markov chain (Feller [40]). In the case when the Markov chain is a random walk on a group, this space is naturally endowed with the action of this group, and there are several equivalent ways to define it (see Furstenberg [42, 43, 44], Kaimanovich, Vershik [60]). If the group acts on a symmetric space, then this action induces an action on a naturally defined geometric boundary of this space. The Poisson-Furstenberg boundary can be viewed in such cases as a probability measure on the geometric boundary, and this measure adds essential information to the understanding of both algebraic and geometric properties of the group. An important feature is that, unlike geometric boundary and unlike some other notions of boundary such as Martin boundary, the Poisson-Furstenberg boundary behaves functorially with respect to the group homomorphisms. This has far-reaching applications, such as Furstenberg’s approach to superrigidity theorems (see Furstenberg, Margulis, [44, 73]). Besides superrigidity, measures on geometric boundaries appear for (not necessarily symmetric) hyperbolic spaces. For example, the measure on the boundary appears in the proof of monotonicity of the hyperbolic volume, where the use of this measure
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is crucial for finite volume non-compact manifolds (Thurston, see the exposition in Gromov [49]).

In the more general context, it happens that there is no natural geometric boundary attached to the group. However, the Poisson-Furstenberg boundary is always well defined, in so far as we fix some probability measure on the group. This is the subject of the present paper. The boundary, regarded as a measure space with the group action, is related to many natural questions in random walks and harmonic analysis, and in the last years it turned out that this space has also applications to the growth of groups.

There are several ways to define the Poisson-Furstenberg boundaries for a random walk on a group. We recall some of the equivalent definitions.

**Definition 1.** Consider two infinite trajectories $X$ and $Y$. We say that they are equivalent if they coincide after some instant, possibly up to the time shift. This means there exists $N, k \geq 0$ such that $X_i = Y_{i+k}$ for all $i > N$. Consider the measurable hull of this equivalence relation in the space of infinite trajectories. The quotient by this equivalence relation is called the Poisson-Furstenberg boundary.

If in this definition we do not allow the time shift, that is, if we say that $X$ and $Y$ are equivalent whenever $X_i = Y_i$ for all $i > N$, then the resulting quotient space is called the tail boundary. For a random walk on a group these two definitions give the same space, while in a more general context of random walks on graphs the tail boundary may happen to be larger than the Poisson-Furstenberg boundary. The Poisson boundary is an interesting notion to study in the more general contexts of Markov chains and random walks, but there is more additional structure on such spaces in the case of random walks on groups. Apart from the above mentioned fact about the tail boundaries, there are many other manifestations of such phenomena. The entropy criterion, for example, which we recall below, does not hold in a more general context of not necessarily homogeneous spaces, such as non vertex-transitive graphs.

A function $F : G \to \mathbb{R}$ is called $\mu$-harmonic, if for all $g \in G$ it holds $F(g) = \sum_{h \in G} F(gh) \mu(h)$. The Poisson-Furstenberg boundary can be equivalently defined in terms of bounded harmonic functions (from the subgroup, generated by the support of $\mu$ to $\mathbb{R}$):

**Definition 2.** The space of all bounded $\mu$-harmonic functions, can be endowed with multiplication: given two bounded harmonic functions $f_1$ and $f_2$, put

$$(f_1 \times f_2)(x) = \lim_{n \to \infty} \sum_x f_1(gx)f_2(gx)\mu^\ast n(x).$$

One can prove that the limit above exists, and that this product is associative. It is easy to check that then $f_1 \times f_2$ is harmonic. Since the limit of bounded harmonic function with respect to the supremum norm is again bounded and harmonic, one concludes that the space of bounded $\mu$-harmonic functions forms a commutative Banach algebra. Its spectrum $\Pi_\mu$ is endowed with probability measure $\nu$, defined by the following equality, which holds for all $f$: $\int \hat{f}(x)d\nu(x) = f(\epsilon)$, where $\hat{f}$ is the Gelfand transform of $f$. $\Pi_\mu$, as a measure $G$-space, is isomorphic to the Poisson-Furstenberg boundary.
In particular, the equivalence of the definitions implies that the group $G$ admits nonconstant bounded harmonic functions with respect to some measure $\mu$, with the support generating $G$, if and only if the Poisson-Furstenberg boundary of the random walk is non-trivial. To see one of the implications observe that for any subset of the boundary, according to the first definition, the probability to hit this set (that is, the probability that the equivalence class of the trajectory belongs to this set) is a harmonic function between 0 and 1, which is non-constant so far as the set, as well as its compliment in the boundary both have positive probability.

For more on different definitions of the Poisson boundary see [60]. For more recent not exhaustive survey see [41].

A harmonic function on a group is a discrete counterpart of a harmonic function on a Riemannian manifold. Given a regular cover $M$, with deck transformation group $G$, there is a symmetric measure $\mu$ on $G$, such that the Poisson-Furstenberg boundary of $G$ can be identified with that of $M$, in particular, $G$ admits bounded $\mu$-harmonic functions if and only if $M$ does. This measure is called Furstenberg discretization or Furstenberg-Lyons-Sullivan discretization [70, 54]. This measure is in general infinitely supported. It has a rapid decay, exponential moments of this measure are finite.

(We do not touch in this paper the questions concerning unbounded harmonic functions, and their relation to the random walk, such as positive harmonic functions and the corresponding Martin boundary, see [84]).

Some of questions one asks about boundaries of random walks are as follows:

- given a group $G$ and a probability measure $\mu$ on $G$, can we say whether the boundary of $(G, \mu)$ is trivial or not?

- If the boundary is non-trivial, can we describe at least some $\mu$-boundaries, that is, some non-trivial quotients of the Poisson-Furstenberg boundary?

- Can one provide a complete description of the boundary $(G, \mu)$?

- Can one obtain some information on the large scale geometry of $G$, granting some information on $\mu$ and the boundary of $(G, \mu)$, such as the triviality/non-triviality of the boundary of this random walk? Such as the description of this boundary?

We recall that if the boundary of the random walk is trivial, then the group, generated by the support of $\mu$, is amenable, so the first question is essentially about amenable groups. It is also known that any amenable group admits a non-degenerate symmetric measure with trivial boundary (Rosenblatt; Kaimanovich, Vershik [76, 59, 60]). For application of this criterion, as well of the generalisation of this criterion for the case of amenable extension [57] see [82, 16, 6].

In many groups symmetric non-degenerate measures, provided by Kaimanovich-Vershik-Rosenblatt criterion, can not be chosen to have finite support. In [30] it is shown that on some groups such measures can not be chosen even in the class of measures with finite entropy. For a probability measure $\mu$, we denote by $H(\mu)$ the entropy of $\mu$, that is, $H(\mu) = -\sum_g \mu(g) \log(\mu(g))$, and we recall the entropy
criterion of boundary triviality. The notion of the entropy of a random walk on a group is due to Avez [3]. The entropy of the random walk $(G, \mu)$ is defined as the limit, as $n$ tends to infinity, of $H(\mu^n)/n$. Here $\mu^n$ denotes the $n$-th convolution of $\mu$, and the limit exists in view of the subadditivity $H(\mu^{n+k}) \leq H(\mu^n) + H(\mu^k)$.

**Entropy Criterion (Avez-Derriennic-Kaimanovich-Vershik).** ([4, 22, 59, 60]) Let $G$ be a countable group, $\mu$ be a probability measure on $G$ such that its entropy $H(\mu)$ is finite. Under this assumption the Poisson-Furstenberg boundary is trivial if and only entropy of the random walk $h(\mu)$ is equal to zero.

This criterion shows that just one number, defined by the $n$-th convolution of $\mu$, gives an answer whether the boundary is trivial or not. The assumption that the entropy is finite is essential. Even if we know exactly the distribution of these $n$-th convolutions (in other words, if we know exactly the abstract distribution after the $n$ steps of the random walk), then we can not in general forget the underlying group structure and say whether the boundary is trivial or not. There exist examples of measures on $\mu$ on $G$, such that the boundary of $(G, \mu)$ is non-trivial, while the boundary of the random walk defined by the inverse measure $(G, \tilde{\mu})$ is trivial (see Kaimanovich [53] and example 6.5 in [60]). It is clear, however, that the distributions after $n$-th step of the random walk are the same for the random walk and the inverse. The entropy criterion tells us that this kind of phenomena can not happen for measures of finite entropy.

Some applications of the entropy criterion are immediate: for example, it tells us that any finitely supported (and more generally, any finite first moment) measure on a group of sub-exponential growth has trivial boundary. In other examples the estimate of the entropy, even understanding whether the entropy is zero or not, can be significantly harder. In the next section we review the recent progress in this direction. In some classes of groups it seems easier to use the part of the entropy criterion that tells us that if entropy is zero, then the boundary is trivial. Though one can in many cases estimate the entropy from below and show that it is positive, in various classes of groups that were studied previously, this was not the only way to see that the boundary is non-trivial. Indeed, in some of the examples one could see directly that there are some non-trivial $\mu$-boundaries [60], in others one was able could construct bounded harmonic functions on the group on on some covers with a given deck transformation group [70]. However, recently one has discovered other classes of groups, where the entropy criterion is so far the only known way to show the non-triviality of the boundary. In the next section we review some examples of this kind, for which we can not answer so far the above mentioned question about $\mu$-boundaries.

An important tool for the complete description of the boundary is Conditional entropy criterion, due to Kaimanovich, which is analogous to the entropy criterion. This criterion tells that if we have some $\mu$-boundary and if we want to check whether this boundary is trivial it is necessary and sufficient (provided we work with measures with finite entropies) to check whether the conditional entropy is zero. In some sense it seems that "the larger the group is", the easier is to use this strategy. In Section 4 we try to give this statement a more precise meaning and we review some known results.
Some applications of boundary are very well studied, for example the already mentioned relation of boundary to the space of harmonic functions. Applications to the growth of groups are more recent. Since entropy criterion was established, it has been known that there is a strong relation between the growth of a group and non-triviality of the boundary. It may seem that the growth of a space or of a group is much easier to determine than triviality/non-triviality of the boundary. However, one can use the boundary behavior of the random walk in some cases (see section 3) as a tool for establishing lower estimates on the growth.

We recall that the word length \( l_S \) with respect to a finite generating set \( S \) of \( G \) is defined as follows. For each \( g \in G \) the word \( l_S(g) \) is the minimum of \( m \), such that \( g \) is equal to the product of \( m \) elements of \( S \) and of their inverses. The \( i \)-th moment \( (i \in \mathbb{R}) \) of a measure \( \mu \) on \( G \) with respect to the word length \( l_S \) is

\[
\sum_{g \in G} \mu(g) l_S^i(g).
\]

**Remark 1.** The rate of escape (or the drift) \( l \) of the random walk \( (G, \mu) \) with respect to some word metric \( l_S \) is defined as the limit of \( L(n)/n \), where \( L(n) \) is the expectation of the distance to the origin after \( n \) steps of the random walk. If \( \mu \) is symmetric and has finite first moment with respect to some (and hence to all) word metrics in \( G \), then the entropy of the random walk is positive if and only if the rate of escape of the random walk \( (G, \mu) \) is positive (It is easy to see that \( h \leq vl \), where \( v \) is the exponential growth rate of the group, (see Guivarc’h [51]), and so it is clear that \( l = 0 \) implies \( h = 0 \). The converse was proved by Varopoulos in [80] for for finitely supported measures and then by Karlsson, Ledrappier in the general case [64]. Furthermore, it is shown in [38] that if \( \mu \) is symmetric and has finite second moment, then there exists \( C > 0 \) such that \( H(\mu^n) \geq C(L(n)/n)^2 \) for all \( n \). (It is shown by Ledrappier [68] [68], that \( h \geq l^2 \) for the Brownian motion on the covering manifold).

Given two function \( f_1, f_2 : \mathbb{N} \to \mathbb{R}_+ \), the notation \( f_1(n) \sim f_2(n) \) means that there exists \( C > 0 \) such that \( f_1(n) \leq Cf_2(Cn) \) and \( f_2(n) \leq Cf_1(Cn) \).

**Remark 2.** The study of the asymptotics of \( L(n) \) was initiated by Vershik and Guivarc’h. It turns out that \( L(n) \) can have various asymptotics [27]. For example, \( L(n) \sim n/\ln(n) \) for any finitely supported random walk on the wreath product of \( \mathbb{Z}^2 \) with a finite group; \( L(n) \sim n^{1-2^{-k}} \) for the \( k \) times iterated wreath product \( \mathbb{Z} \wr (\mathbb{Z} \wr \cdots \wr \mathbb{Z}) \). For further examples of evaluation and estimates of \( L(n) \) see [27], [38], Yadin [85], and also Corollaries 1 and 2 in Section 3 and the remark after Theorem 6 in Section 5.

It is not known so far whether any function \( f(n) \) between \( \sqrt{n} \) and \( n \), with some regularity on its growth, is asymptotically equivalent to \( L(n) \) for a simple random walk on some finitely generated group \( G \). We also mention that upper bounds on \( H(n) \) can be relevant for Liouville type theorems on the growth of unbounded harmonic functions (see [38]).
2. Applications of entropy criterion

The entropy criterion can be used to show that the simple random walks on polycyclic groups and on solvable Baumslag-Solitar groups have trivial boundaries. See also [26] for triviality of the boundary for some random walks on iterated wreath products of $\mathbb{Z}$ and $\mathbb{Z}^2$.

A very interesting class of examples was discovered recently by Bartholdi and Virag [10], who studied a group that was defined earlier by Grigorchuk and Żuk. Using a notion of a "self-similar" random walk, they have shown that this group admits a finitely supported measure with zero rate of escape. Originally in their paper they used some special metric on this group, which is not a word metric, and later the argument was simplified by Kaimanovich [61] who works with $H(\mu^\infty)$ instead of the rate of escape and shows that the entropy of the random walk is zero. It turned out that this argument can be applied to wider classes of groups acting on rooted trees. See [11] for the case of groups generated by bounded automata and [2] for a more general case of groups generated by so called linear activity automata (it is shown by Sidki in [77] that a group generated by a polynomial activity automaton never contains a non-Abelian free subgroup, and it is an open question whether all such groups are amenable). An interesting feature of the above mentioned examples is that the non-vanishing of the rate of escape or the entropy of a non-degenerated random walk is zero is used to show that the groups under consideration are amenable. Thus, random walks and Kesten criterion help to understand in these examples whether the group is amenable. In all previously known amenable finitely generated group there is some known sequence of Folner sets.

Now we recall some examples of random walks with non-trivial boundaries. The simplest class of examples of amenable groups such that the simple random walks have non-trivial boundaries are wreath products $\mathbb{Z}^d \wr A$ (that is, semidirect products of $\mathbb{Z}^d$ with $\sum_{x \in \mathbb{Z}^d} A$, with $\mathbb{Z}^d$ acting by shift on the index set), $d \geq 3$ (Kaimanovich, Vershik, [60]). To see that the boundary of the simple random walk is non-trivial, one observes that the projection of the random walk to the base group $\mathbb{Z}^d$ is transient, and that therefore for all $x \in \mathbb{Z}^d$ the coordinate $a_x$ stabilizes along infinite trajectories of the random walk. This argument has generalizations in several contexts, where its application is less straightforward. Quotients of the Poisson-Furstenberg boundaries for certain groups acting of rooted trees, that we describe in Section 3, is reminiscent of this "lamplighter boundary" for wreath products. We recall also, that Kaimanovich [62] has shown, that a simple random walk on the Thompson group $F$ has non-trivial boundary. Kaimanovich has observed that if a group acts on a line by piecewise-linear mappings with finite number of pieces, then the boundary is non-trivial whenever the orbits of the action are transient (since for such actions the ratio of the left and right derivatives at a given point stabilizes along infinite trajectories), and he has proved that for the group $F$ these orbits are indeed transient. It would be interesting to understand the boundary behavior of random walks on the more general groups of diffeomorphisms of the interval for a) simple random walks; b) for not necessarily finitely supported random walks. It is a long standing question whether Richard Thompson group $F$ is non-amenable. If
it turns out to be amenable, the these questions become especially interesting.

Now we return to the wreath products and recall some additional properties of the boundaries of random walks on these groups. A transience argument, similar to the argument used in finitely-supported case in [60], shows that any non-degenerate random walk with finite first moment has non-trivial boundary [55]. Another argument, based on an entropy estimation, is introduced in [30]. It consists of subdividing the space of trajectories of length $n$ into conditional subspaces, such that there exists a subset of measure at least $p$, ($p > 0$ is a constant not depending on $n$), with the following properties: the trajectories, belonging to the same conditional event inside this subset all have the following form. There exits a sequence $n_1^{(n)}, n_2^{(n)}, \ldots, n_k^{(n)}$, depending on each conditional event, such that $k_n \geq Cn$, where $C$ is a positive constant not depending on $n$. The increments of the trajectory at times $t$ others then $n_i^{(n)}$, $1 \leq i \leq k_n$ are fixed for a given conditional event. For each time instant $t$, $t = n_i^{(n)}$ (for some $i$) the increments take two possible values. All $2^{n_k}$ trajectories in each given conditional event visit at moment $n$ different elements of the group. The time instants $n_1^{(n)}, n_2^{(n)}, \ldots$ correspond to visits of distinct points by the projection of our random walk to some space. In the case of wreath products $\mathbb{Z}^d \wr A$ this space is $\mathbb{Z}^d$.

If the random walk admits such partition into conditional events, then the inequality between entropy and mean conditional entropy implies that the entropy of the random walk is positive. It might seem that the assumption is essentially stronger then the positivity of entropy, but it is shown in [30] that it can applied to many classes of groups. Moreover, it is not clear whether there are any obstructions for this type of entropy estimates: does there exists a simple random walk $(G, \mu)$, having non-trivial boundary, and not admitting families of conditional events, satisfying these properties?

Such question can be viewed as a probabilistic (entropic) counterpart of the following still open question, raised by Rosenblatt in [76]: does any group $G$ of exponential growth admit a Lipschitz imbedding of the infinite binary tree?

The argument, applied to the wreath products, shows that any non-degenerate random walk of finite entropy on wreath products $\mathbb{Z}^d \wr A$, $d \geq 3$, #A $\neq 1$, has non-trivial boundary. The same conclusion holds for the free metabelian group on $d$ generator: $\langle g_1, g_2, g_d | uw = wu, u, w \in [G, G] \rangle$. Another series of examples, studied in [30] is as follows. Consider a finitely presented group $B_d$, defined by the following generators and relations

$$B_d = \langle a, s_i, t_j | a^{s_i} = aa^{s_i}, [s_i, s_j] = [t_i, t_j] = [s_i, t_j] = e, [a^u, a^w] = e \rangle,$$

where $i, j$ in the presentation take the values between 1 and $d$, and $u$ and $w$ are any words in $s_i$ and $t_j$. The group $B_d$ is a subgroup of $\text{GL}(2, \mathbb{Z}(X_1, \ldots, X_d))$. It is a metabelian (that is, solvable of solvability length 2) group, and its subgroup generated by $s_i$ and $a$ is isomorphic to the wreath product of $\mathbb{Z}^d \wr \mathbb{Z}$. The groups $B_d$ are particular cases of a more general construction due to Baumslag, that assures that any finitely generated metabelian group can be imbedded into finitely presented metabelian group. However, there is no known relation in general between triviality of the boundary of random walks on a subgroup and triviality of
the boundary of random walks on the ambient group. There are particular cases, where such relation does exist. For example, this relation is well known in the case when the subgroup is recurrent for our random walk, that is, if the random walk $(G, \mu)$ returns to the subgroup $H$ infinitely many often. In this case one considers the probability measure $\mu'$ on $H$, such that for any $h \in H$ the probability $\mu'(h)$ is equal to the probability that the random walk visits $h$ at the instant of its first return to the subgroup. One shows that there is a canonical measure preserving bijection between the boundary of $(G, \mu)$ and that of $(H, \mu')$. A recent result of Malyutin, Vershik [72] shows that for any group $G$, containing a free subgroup, and any simple random walk $\mu$ on $G$, the boundary of this free subgroup is a $\mu$-boundary for the random walk $(G, \mu)$. (These particular cases are not relevant to random walks we discuss). In general, it is not known whether non-triviality of some (all) simple random walks on the subgroup implies the non-triviality of some (respectively all) simple random walks on a group, containing this subgroup. Thus the non-triviality of the boundary for simple random walks on the wreath products did not help to prove the non-triviality of the boundary of random walks on $B_d$. Entropy estimates from [30], applied to $B_d$, show that for $d \geq 3$ any simple random walk on $B_d$ (and, more generally, any non-degenerate random of finite entropy) has non-trivial boundary.

Since Furstenberg discretization has finite entropy and since every finitely presented group, in particular, our $B_d$, serves as the fundamental group of a compact manifold $M$ of a given dimension $> 3$, the above implies the following

**Theorem 1.** There exists a compact Riemannian manifold $M$, such that its fundamental group is amenable and such that its universal cover is not Liouville (that is, this universal cover admits non-constant bounded harmonic functions).

**Question 1.** What is the Poisson-Furstenberg boundary for the simple random walks on groups $B_d$?

It is not clear even how to describe any non-trivial quotient of the boundary for these random walks.


It is known that any random walk on a finitely generated group of polynomial growth has trivial Poisson-Furstenberg boundary. Indeed, it is shown by Dynkin and Malyutov in [25] that this statement, generalizing the classical Choquet-Deny theorem for Abelian groups (Blackwell, [12]), holds for any finitely generated nilpotent group (see also [73] for description of all positive harmonic functions on nilpotent groups). By Polynomial Growth Theorem of Gromov [48] any group of polynomial growth is a finite extension of a nilpotent one. This can be used to show
that the triviality of the boundary for nilpotent groups implies the triviality of the boundary for any measure on a group of polynomial growth. Now let $G$ be a group of subexponential growth. That is, either $G$ is of polynomial growth and is virtually nilpotent, or it has growth strictly between polynomial and exponential.

If we suppose that the measure $\mu$ on $G$ has finite first moment, then in view of the entropy criterion the boundary is trivial. The question was whether a counterpart of Choquet-Deny theorem holds for any measure on a group of subexponential growth, that is, whether the condition to have finite first moment in the above mentioned statement is not essential. A negative answer is given in [31], where it is shown that some among Grigorchuk groups of intermediate growth admit a measure with non-trivial boundary. Moreover, on some of these groups this measure can be chosen to have finite entropy.

The idea of the construction of such measures and of the proof that the boundary is non-trivial is as follows. Given a group, acting on a rooted tree, we consider the action on the boundary of the tree and an orbit of a point $x$ of the boundary under this action. In [31] we used an equivalent language of groups, acting by permutation of the interval (where the points of the interval $(0, 1]$ are written as numbers in the $k$-ary numeral system, which correspond to the points of the $k$-regular rooted tree together with its boundary). The main focus in that paper is on Grigorchuk groups and their close generalizations. In this situation $x$ could be chosen (in the terminology of actions on a rooted tree) to be the point of the boundary, corresponding to the right most ray of the tree. We say that the action of $G$ on the interval $(0, 1]$ verify the strong condition (*) if the following holds. For any $g \in G$, $x, y \in (0, 1]$ such that $g(x) = y$ and any $\delta > 0$ there exist $\epsilon > 0$ such that $g((x - \epsilon, x]) \subset (y - \delta, y]$. There exists a finite generating set $S$ of $G$ such that for any $s \in S$ and $x \in (0, 1]$ satisfying $x \neq 1$ or $s(x) \neq 1$ there exist $a \in \mathbb{R}$ and $\epsilon > 0$ such that $s(y) = y + a$ for any $y \in (x - \epsilon, x]$. The standard action on the interval of any Grigorchuk group satisfies the strong condition (*).

Below we use the language of actions on trees, that seems slightly more adequate for some more general questions we want to address. Let $G$ be a group acting on a rooted trees. For all $y$ on the orbit of $x$ one chooses a mapping $T_{yx}$ defined from a left neighborhood of $y$ to a left neighborhood of $x$ in such a way that for all $x, y, z$ the mapping $T_{zy}T_{zx}$ coincides with $T_{zx}$ in some left neighborhood of $y$. Mappings $T_{yx}$ allow us to multiply germs at different points, and we consider then the group of germs, generated by all germ($g, y$), where $g \in G$ and $y$ is on the orbit of $x$. We say that the action on a tree satisfies the strong condition (*) if the corresponding action on the interval satisfies this condition.

**Theorem 2.** Let $G$ be a group acting on the rooted tree, such that the action satisfies the strong condition (*) and suppose that there exists a subgroup $H$, such that the group of germs of $H$ is not equal to the group of germs of $G$ and such that the orbit of $x$ under the action of $H$ is infinite. Then $G$ admits a measure with non-trivial boundary.

In [31] it was assumed in Theorem 2 above that the group of germs of $G$ is finite, but this condition can be easily dropped. One may check that the assumption of
this theorem are verified for some of Grigorchuk groups. Moreover, on some of these
groups one can additionally show that the constructed measure can be chosen to
have finite entropy. Thus we get

**Theorem 3.** i) There exist groups $G$ of subexponential growth admitting probability
measures $\mu$ with non-trivial Poisson boundary.

ii) Moreover, there exist groups $G$ of subexponential growth admitting probability
measures $\mu$ of finite entropy such that the entropy $h(\mu)$ of the random walk $(G, \mu)$
is positive.

Theorem 2 can be applied to groups acting on rooted trees, the growth of
which can be exponential or intermediate. It seems the most interesting that it
can be applied to a large range of groups of intermediate growth. We want to
stress however, that there are groups where it can not be applied and where we
still do not know the answer to the question: does this group admit a measure with
non-trivial boundary? In particular, this remains unknown for the first Grigorchuk
group, which is the most well studied among groups of intermediate growth.

To prove Theorem 2, one constructs a measure $\mu$ such that its support belongs
to the union of the subgroup $H$ with some finite set in $G$ and such that the induced
random walk on the orbit is transient. One shows then that germ$(g, x)$ modulo
the group of germs of $H$ stabilizes along infinite trajectories of the random walk.

The condition (*) in the way it is defined [31] is well suited for Grigorchuk
groups, considered in that paper. In last years many new interesting examples
of groups acting on rooted trees have been studied, for which this condition does
not hold. It seems that this assumption in the theorem above can be very much
weakened, and it is interesting to understand what is the optimal condition.

Our main motivation Theorem 2 is the construction of infinitely supported
measures with non-trivial boundary (Theorem 3). However, a particular case of
Theorem 2 above is when the orbital Schreier graph of $H$ is transient. In this case
the theorem shows that the Poisson-Furstenberg boundary of the simple random
walk on $G$ has non-trivial boundary. Recently Bondarenko [14] has shown that if
$G$ is generated by a bounded automaton, then the orbital Schreier graph of $G$ is
recurrent. It is known that such groups can be imbedded in a group, admitting a
simple random walk with trivial Poisson-Furstenberg boundary. It seems that the
assumption (*) in the corollary can be much weakened.

**Question 2.** Can the the criterion from [31] be extended to provide a general
criterion for recurrency/transiency of orbital Schreier graphs for groups acting
on rooted trees?

To have a sufficient condition for the Schreier graph being recurrent, we have
exclude cases such as $Z^d$, $d \geq 3$, which act on a rooted trees and have trivial
boundary, but it is seems that there could be criteria, much more general then
those explained in [31], in terms of triviality of the boundary.

**3.1. Application to growth.** Let $G$ be a finitely generated group and $S$ be
a finite generating set of $G$. The growth function $v_{G,S}(n)$ is the number of elements
of $G$ that can be written as a product of at most $n$ elements of $S$ and their inverses. It is shown in [31] that some measures with non-trivial boundaries on groups of intermediate growth can be used to obtain lower bounds on the growth of these groups. They are used in [31] to obtain the following bounds on the growth of certain Grigorchuk groups

$$\exp(n/\log^{2+\epsilon}(n)) \leq v_{G,S}(n) \leq \exp(n/\log^{1-\epsilon}(n)),$$

for all sufficiently large $n$. Here the lower bound is essentially due to Grigorchuk. The lower bound follows from the fact, that the group $G$ admits a measure $\mu$ with non-trivial boundary, with a certain control on the decay of $\mu$. A generalization of this idea is introduced in [33], where we provide new lower bounds for the growth of groups of the form $\exp(n^\gamma)$.

Another application in [33] provides lower bounds for the escape $L(n)$ of random walks on certain groups, acting on rooted trees. The strategy is as follows. Given a group $G$, acting on a rooted tree, construct another auxiliary group $G_2$, $G \in G_2$, such that the group of germs of $G_2$ is larger than the group of germs of $G$. In this situation the upper bounds on growth of $G_2$ provide lower bounds for the asymptotic behavior of $L(n)$ for random walks on $G$. We introduce in [33] the critical constant $c_{RT}(G,H)$ of a subgroup $H$ in a group $G$. This constant is defined as $\sup \beta$, where the supremum is taken over all $\beta$, for which there exists a random walk on $G$, of finite $\beta$-moment, such that the induced random walk on $G/H$ is transient. Suppose that the action of the auxiliary group $G_2$ satisfies the strong condition (*) and that the growth function of $G_2$ is bounded from above by $\exp(n^\gamma)$. One proves that in this case the critical constant of the stabilizer of 1 in $G$ is at most $\gamma$. The proof uses the Poisson boundary argument similar to the proof of theorem 2. On the other hand, one observes that if $L(n) \leq Cn^2$, then $c_{RT}(G,H) \geq 1/(2\xi)$ for any finite index subgroup $H$ in $G$. Moreover, if $c_{RT}(G,H) < 1/(2\xi)$, then $\sum_{n=1}^{\infty} L(n)n^{-(1+\epsilon+\xi)} = \infty$, for some $\epsilon > 0$. Applying this for $H$ which is equal to the stabilizer of 1 in $G$, we conclude that the asymptotics of the escape of any simple random walk $(G,\mu)$ satisfies $L_{G,\mu}(n) \geq n^{1/(2\gamma)}$ for infinitely many $n$. For example, let $G$ be the first Grigorchuk group. In this case one is able to construct the auxiliary group $G_2$, with the growth at most $\exp(n^\gamma)$, where $\gamma = \log 2/\log 2/X$ and $X$ is the positive solution of the equation $X^3 + X^2 + X - 2$. (For the first Grigorchuk group such upper bound on the growth function is due to Bartholdi [9], and a similar argument works also for our group $G_2$). We have $\gamma < 0.768$. This implies

**Corollary 1.** [33] For any simple random walk on the first Grigorchuk group

$$\sum_{n=1}^{\infty} L_{G,\mu}(n)n^{-(1.65)} = \infty, \text{ and } L_{G,\mu}(n) > n^{0.65} \text{ for infinitely many } n.$$

It is proved by Grigorchuk that some of his groups are close to the first Grigorchuk group on one scale, and they are close to subgroups in direct sum of several copies of a solvable group $H$ of exponential growth on the other scale. For this group $H$ and for any symmetric finitely supported measure $\mu$ on $H$ one can check that $L_{H,\mu} \leq C_1 \sqrt{n}$, and this can be used to obtain the following corollary.
Corollary 2. There exists a Grigorchuk group $G$, such that a simple random walk on $G$ satisfies $\lim\sup(\log L_{G,\mu}(n))/n \geq 0.65$ and $\lim\inf(\log L_{G,\mu}(n))/n \leq 1/2$.

It is known (Lee, Peres [69]) that if $G$ is a finitely generated group, and $\mu$ is a symmetric finitely supported measure, such that its support generates $G$, then $L_{G,\mu}(n) \geq C\sqrt{n}$, for some $C > 0$ and all $n$.

Question 3. Let $G$ be a finitely generated group. Suppose that $L_{G,\mu}(n) \leq C\sqrt{n}$, where the measure $\mu$ is such that its support generates $G$. Can the growth of $G$ be intermediate?

There is no Grigorchuk group for which we know precisely the asymptotics of the growth function. And the estimates, obtained using random walks as explained above, provide in a sense the best known examples, where discrepancy between the upper and lower bounds is not too large. It would be very interesting to obtain more information on possible functions, that can be realized as the growth function of some groups. Grigorchuk has shown that there are groups with arbitrarily fast subexponential growth (more precisely, Grigorchuk shows in [45] that among his groups there are groups such that their the growth is minorized along a subsequence by a given subexponentially growing function, and essentially the same argument [32] shows that by taking a direct sum of two Grigorchuk groups we obtain a growth function, that is minorized by a given subexponential function for all sufficiently large values of $n$). A natural question would be: can any sufficiently fast growing subexponential function be realized as a growth function of some group? In particular, we want to know: does there exist $a < 1$ such any function $f \geq \exp(n^a)$ as a growth function?

It is even more challenging to construct groups of super-polynomial growth with the smallest possible growth. A conjecture due to Grigorchuk [46] states that any super-polynomial growing function is bounded from below by $\exp(nb)$, for some $b > 0$ (the strong form of this conjecture states that we can take $b = 1/2$). It is tempting to understand better the possible applications of the boundary theory for the class of groups of small intermediate growth. See [47] for other question concerning the growth of groups and, in particular, of Grigorchuk groups.

4. Complete description of Poisson-Furstenberg boundaries

The complete description of the Poisson-Furstenberg boundary has been known for the following finitely generated groups (under certain conditions on the decay of the probability measure defining the random walk):

- discrete subgroups in semi-simples Lie group (Furstenberg [44] for a particular case of an infinitely supported measure, "Furstenberg approximation", Ledrappier [67] for a more general class of measure on discrete subgroups of $SL(d,\mathbb{R})$, Kaimanovich [56] for a more general class of measures on discrete
subgroup in an arbitrary semi-simple Lie group); see also Schapira [?] and [17];

- free groups (Dynkin, Malyutov [25] for random walks, with the defining measure supported on standard generators, Derriennic [23] for measures with finite support), more generally for hyperbolic groups (Ancona [1] for measures with finite support, Kaimanovich [56] for measures of finite entropy and with finite logarithmic moments; see also Ballman Ledrappier [8]; for the question whether a given measure on the hyperbolic boundary can be realized as the hitting measure of a certain random walk see [19]),

- Coxeter groups (follows from Karlsson, Margulis [65], see Theorem 6.1 in [63] for an explanation),

- groups with infinitely many ends (Woess [83] for finitely supported measures, [56] for a more general class of measures),

- the mapping class group (Kaimanovich, Masur [58]) and braid groups (Farb, Masur [39]).

We would like to stress that for some of the above mentioned groups the boundary is described in terms of the space on which the group acts. It could be important and in some situations it seems to be harder to describe the boundary in more algebraic terms (see Vershik [81] for the statement of the problem and Malyutin, Vershik [72] for the results in this direction, including the stability of the so-called Markov-Ivanovsky normal form for random walks on braid groups).

- Wreath products of free groups with finite groups (Karlsson, Woess [66]),

- certain classes of groups acting by diffeomorphisms on a circle (Deroin, [21]).

For some classes of groups it is easier to identify the boundary for certain non-symmetric random walks, rather than for symmetric ones. It was done for

- random walks on wreath product $\mathbb{Z}^d \wr B$ with a non-zero drift of the projection on $\mathbb{Z}^d$ [56],

- for random walks on solvable Baumslag-Solitar groups with a non-zero drift of the projection on $\mathbb{Z}$ [55], and, more generally, for such random walks on the group of rational affinities [15]. In the last two examples simple random walks have trivial boundary.

It was asked in [60] whether the "space of limit configurations", described in Section 2, provides a complete description of the Poisson-Furstenberg boundary in wreath products $\mathbb{Z}^d \wr A$ ($d \geq 3$). A positive answer for $d \geq 5$ is given in [35], where we prove

\textbf{Theorem 4.} \textit{Let } $A = \mathbb{Z}^d$, $d \geq 5$, $\#B \geq 2$. If $\mu$ is a measure on $C = A \wr B$, that the support of $\mu$ generates $C$ as a group, the third moment of $\mu$ is finite and the projection of $\mu$ to $\mathbb{Z}^d$ is centered, then the Poisson-Furstenberg boundary is equal to the space of limit configurations.
We hope that the argument in [35] can be extended also to the case of $d = 3$, $d = 4$.

The theorem above holds in general, without the assumption that the projection of $\mu$ to $\mathbb{Z}^d$ is centered. If this projection is not centered, than the projected random walk on $\mathbb{Z}^d$ has positive drift. As we have already mentioned, for measures such that the projection has positive drift, the result is due to Kaimanovich. Another partial case of the statement of the theorem that was known previously is due to James and Peres, who have shown in [52] that the number of visits of points of the base provides a complete description of the Poisson-Furstenberg boundary of a certain measure on the semigroup $\mathbb{Z}^d \wr \mathbb{Z}^+$. The Poisson boundary of particular random walks on wreath products of $A$ with $\mathbb{Z}^+$ are equivalent to the exchangeability boundary of the projection random walk on $A$ (for definition of the exchangeability boundary, its properties and questions about this boundary see [13, 24, 55, 52, 35]).

We hope that the argument in [35] can be extended also to the case of $d = 3$, $d = 4$.

A similar idea to that in the proof of Theorem 4 leads to description of the boundary for free metabelian groups ([35]). It can be applied also to other groups with some resemblance to wreath products, such as for example extensions by a finitely generated group $A$ of the finitary symmetric group on elements $A$.

In previous work that provided complete description of the boundary (see [44, 56, 67, 58, 65] and other above mentioned results), there is a natural candidate for the Poisson-Furstenberg boundary, and, moreover, there is a natural guess along which "directions" the trajectories converge to the limit point in this boundary. The main work is then to estimate conditional entropy (in many cases this can be done using Ray Criterion, though in some situations: modular group, a measure on a word-hyperbolic group without first moment, it is easier to work with Strip Criterion). One of the difficulties in proving Theorem 4 is that for wreath products, though there exists a natural and easy to describe candidate for the Poisson-Furstenberg boundary (lamplighter boundary), it is however not straightforward even to guess how the trajectories converge to the points of this boundary. The first step in the proof is to use the geometry and connectivity properties of the support of the limiting lamplighter configuration in order to "guess" approximatively which points the trajectory visit at a certain time instants; the second step is to use this as a "ray approximation", to estimate conditional entropy and to prove, that "lamplighter boundary" is indeed the Poisson-Furstenberg boundary of the random walk under consideration.

5. Different scales of amenability. Asymptotic invariants related to boundaries

Consider a symmetric non-degenerate probability finitely supported measure $\mu$ on $G$. In the sequel we call random walk, defined by such measures, simple. As we have already mentioned, the boundary triviality of the random walk $(G, \mu)$ implies the amenability of $G$. For some amenable $G$ the boundaries of $(G, \mu)$ is trivial, while
for others such boundaries can be non-trivial. It is as an open question whether the triviality of the boundary can depend on the choice of a simple random walk. For other questions related to dependence of entropy on the choice of defining measure see [36] and [37].

The fact that the Poisson-Furstenberg boundary of a simple random walk on $G$ is trivial can be viewed as a strengthening of the fact that $G$ is amenable.

Recall that a group $G$ is said to be amenable, if it admits a finitely additive non-negative measure $\nu$ defined on all subsets of $G$, which is invariant under left translations and which has total mass one.

Kesten criterion of amenability says that a finitely generated group $G$ is amenable if and only for some (and if and only if for all) non-degenerate finitely supported symmetric random walks on $G$ the decay of the probability to return to the origin is subexponential. Another criterion is in terms of isoperimetric inequalities. Let $S$ be a finitely generating set $S$. Given a subset $V \subset G$, its boundary $\partial_S V$ with respect to $S$ is \{ $v \in V : 3s \in S : vs \not\in V$ \}. By Følner criterion of amenability a finitely generated group is amenable, if there exists a sequence of finite subsets $V_n$ such that $|\partial_S V_n|/|V_n| \to 0$, as $n$ tends to $\infty$. Here $|V|$ denotes the cardinality of the set $V$. The sequence $V_n$ is called a Følner sequence, and the sets $V_n$ are called Følner sets.

Given an amenable group $G$ and a finite generating set $S$, the Følner function $Fol_{G,S}(n)$ is defined as the minimum of cardinality of $V$, where the minimum is taken over all finite subsets of $V$ of $G$, such that the cardinality of the boundary of $V$ with respect to the word metric $l_{G,S}$ is at least $n$ times smaller than the cardinality of $V$. It is easy to see that if the group admits a sequence of Følner sets, then it admits an invariant mean: given a function on $G$, it suffices to consider the average value of this function for each Følner set, and then take the limit of this average, as $n$ tends to $\infty$, along any non-principal ultrafilter. For for a survey of equivalent definitions of amenability, see [18, 79].

Understanding the asymptotics of Følner function (in other words, understanding optimal isoperimetric inequality), in particular, obtaining lower bounds for Følner function is a question, related to large-scale geometry of groups. The study of Følner function was initiated by Vershik, who conjectured that $\mathbb{Z} \wr \mathbb{Z}$ provides an example of a group, with super-exponentially growing Følner function and asked whether the asymptotics of this function is $n^\alpha$. Følner function of nilpotent groups were studied by Pansu, who proved the first asymptotically optimal isoperimetric inequality for a nilpotent, non virtually Abelian group. Later Varopoulos has shown that for virtually nilpotent group of growth $n^d$ the Følner function is asymptotically equivalent to $n^d$. His result was generalized by Coulhon and Saloff-Coste in [20], who have proved that for any group $G$ the Følner function is asymptotically not less then the growth function of $G$: there exists $C$ such that $Fol_{G,S}(Cn) \geq v_{G,S}(n)$.

Pittet and Saloff-Coste [74] have shown the the Følner function of $\mathbb{Z}^d \wr \mathbb{Z}/k\mathbb{Z}$, $d > 2$ is super-exponential (but their lower bound for the Følner function for these groups was not asymptotically optimal). The question of Vershik is answered in [28], where we prove the following more general
Theorem 5. There exists $C > 0$ such that the following holds. Let $A$ and $B$ be two finitely generated group, $B$ contains at least two elements. Let $S_A$ and $S_B$ be finite generating sets of $A$ and $B$ respectively, and $S$ be the generating set of $A \wr B$, corresponding to the union of $S_A$ and $S_B$. Then

$$\text{Føl}_{A\wr B,S}(n) \geq C \text{Føl}_{B,S_B}(Cn)^{C \text{Føl}_{A,S_A}(Cn)}.$$ 

Under mild assumption on regularity of $\text{Føl}_A(n)$, the theorem provides asymptotically optimal lower bound for Følner function of the wreath product. Thus we obtain the first explicit examples of super-exponential asymptotics of $\text{Føl}_{G,S}(n)$, for example, it shows $\text{Føl}_{\mathbb{Z}\wr\mathbb{Z}} \sim n^n$, $\text{Føl}_{\mathbb{Z}d\wr\mathbb{Z}/k\mathbb{Z}} \sim \exp(nd)$. The theorem also implies that $m$ times iterated exponent (for any $m \geq 1$) is a Følner function of some group.

Wreath products and groups resembling wreath products (see Gromov [50]) are so far the only known examples of groups with super-exponential Følner function, where we know the asymptotics of this function.

However, usually it is much easier to obtain a not necessarily optimal upper bound for the Følner function, that is, to produce a not necessarily optimal sequence of Følner sets in groups. For example, it is not difficult to see that if $G$ is a group of subexponential growth, then some subsequence of balls $B_{G,s}(n_i)$ and corresponding spheres $\text{Sph}_{G,s}(n_i)$ satisfies $\#\text{Sph}_{G,s}(n_i)/\#B_{G,s}(n_i) \to 0$, that is, this subsequence of balls is a Følner sequence. This shows, that though asymptotic geometry and the forms of balls in the intermediate growth case is complicated and quite different from polynomial growth case, there is a certain common properties of such groups and Abelian and nilpotent groups with respect to isoperimetry.

Conjecturally, there could be also some algebraic manifestation of the fact, that groups of intermediate growth, however intriguing they may be, share some common properties with nilpotent groups. We recall a question due to Grigorchuk: whether all infinite simple groups have exponential growth? All Grigorchuk group act on rooted trees, and hence they are residually finite, and thus not simple. Not all groups of intermediate growth are residually finite: there exist central extensions (finite and infinite) of first Grigorchuk groups that have intermediate growth and that are not residually finite [29]. One can show (see Bajorska Macedonska [7]) that one of the two following statements hold: either any group of intermediate growth in an extension of a residually group of intermediate growth; or there exist simple groups of intermediate growth. Indeed, let $G$ be a group of intermediate growth and let $R$ be the intersection of finite index subgroups in $G$. If $G/R$ has super-polynomial growth, then this group is a residually finite group of intermediate growth. If the growth of $G/R$ is polynomial, one proves that $R$ is a finitely generated group, and concludes that $R$ is a group of subexponential growth without subgroups of finite index. Take a simple quotient of $R$. It is clear that this quotient is an infinite group of subexponential, and hence of intermediate growth.

Another question is due to Grigorchuk and Pak: does an infinite group of subexponential growth always admit two infinite commuting subgroup? As to the first question, all infinite simple groups, known until now, are non-amenable. It is worth mentioning, that even the following weaker statement is unknown for the class of infinite groups of subexponential growth (and one might ask the same
question for the larger class of groups, admitting simple random walk with trivial Poisson-Furstenberg boundary, and there are no known counterexamples even among amenable groups):

**Question 4.** Let $G$ be an infinite group of subexponential growth. Does there always exist an infinite subgroup $H$, which has infinite index in $G$?

One could be inclined to say, that groups of subexponential growth are amenable in a very strong sense. For all Grigorchuk groups it is known additionally, that the Følner function of any of these groups is asymptotically bounded by $\exp(n^A)$. Moreover, $A$ can be taken equal to 2 using to a self-similar random walk argument: see [61], where it is explained that first Grigorchuk group has a self-similar measure with additional weight $1/2$ at the identity. This measure is supported on standard generators of the group, and it is symmetric. A similar argument shows that all Grigorchuk groups have a sequence of self-similar symmetric measures (that is, the measures are similar to the corresponding measure on a group of shifted measures), also with additional weight $1/2$ at the identity. The latter fact can be used to show that any Grigorchuk group, the entropy of the random walk, defined by the above mentioned measure, satisfies $H(n) \leq Cn^{1/2}$. This implies that for this (and hence also for any other simple random walk, see Pittet, Saloff-Coste [75]) on any Grigorchuk group, the probability to return to the origin satisfies $p_n(e,e) \leq \exp(-Cn^{1/2})$. The latter implies that Følner function of any of Grigorchuk groups is bounded from above by $\exp(Cn^2)$, for some $C > 0$. The main result of [34] shows, however, that Følner function of a group of subexponential growth can be arbitrarily large, that is

**Theorem 6.** Given a function $f : \mathbb{N} \to \mathbb{R}$, there exists a group $G_f$ of intermediate growth such that

$$\text{Fol}_{G_f,S}(n) \geq f(n)$$

for all $n$.

The group $G_f$ in this theorem can be chosen to be a torsion-group. Alternatively, it can be chosen to be a group without torsion. It would be interesting to understand in more detail growth and isoperimetry of such groups.

Another application of the construction from [34] is the existence of finitely generated group $H$ of intermediate growth such that for any $\mu$ on $G$ the escape satisfies $\limsup(\log L_{H,\mu}(n))/n = 1$, $\liminf(\log L_{H,\mu}(n))/n \leq \gamma$, for some $\gamma < 1$. (In terminology of [34] the group $H$ is equal to an appropriate "piecewise automatic group" of the first Grigorchuk group with a non-amenable group). Moreover, one can use the construction from [34] to produce examples of groups $H$ such that for any simple random walk on $H$ it holds $\limsup(\log L_{H,\mu}(n))/n = 1$, $\liminf(\log L_{H,\mu}(n))/n = 1/2$.

The groups in Theorem 6 provide the first examples of groups with very large isoperimetry, such that simple random walks on these groups have trivial boundary.

**Question 5.** Can such phenomenon occur for elementarily amenable group?
**Question 6.** What is the asymptotically largest possible Følner function for a solvable group, admitting a simple random walk with trivial boundary?

As we have already mentioned, Theorem 6 shows that there is no upper bound for Følner function for groups with trivial boundary of simple random walks. If we suppose on the contrary, that the boundary of a simple random walk is non-trivial, Følner function of $G$ can not be too small. Indeed, by the entropy criterion we know that the growth of $G$ under this assumption is exponential, and by Coulhon Saloff-Coste isoperimetric inequality this implies that Følner function is asymptotically at least exponentially growing.

**Question 7.** Suppose that a simple random walk on $G$ has non-trivial boundary. What is the asymptotically smallest possible Følner function of $G$?

**References**


